Existence and regularity of propagators for multi-particle Schrödinger equations in external fields

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Abstract

We consider Schrödinger equations for N number of particles in (classical) electro-magnetic fields which are interacting each other via time dependent inter-particle potentials. We prove that they uniquely generate unitary propagators $\{U(t,s), t,s \in \mathbb{R}\}$ on the state space $\mathcal{H}$ under the conditions that fields are spatially smooth and do not grow too rapidly at infinity so that propagators for single particles satisfy Strichartz estimates locally in time and, that local singularities of inter-particle potentials are not too strong that time frozen Hamiltonians define natural selfadjoint realizations in $\mathcal{H}$. We also show, under very mild additional assumptions on the time derivative of inter-particle potentials, that propagators possess the domain of definition of the quantum harmonic oscillator $\Sigma(2)$ as an invariant subspace such that, for initial states in $\Sigma(2)$, solutions are C^1 functions of the time variable with values in $\mathcal{H}$. New estimates of Strichartz type for propagators for N independent particles in the field will be proved and used in the proof.

1 Introduction

We consider N number of d-dimensional non-relativistic quantum particles of masses $m_j > 0$ and charges $e_j \in \mathbb{R}$, $0 \le j \le N$. We denote the position of j-th particle by $x_j = (x_{j1}, \dots, x_{jd}) \in \mathbb{R}^d$, dx_j is the d-dimensional Lebesgue measure and $\underline{x} = (x_1, \dots, x_N) \in \mathbb{R}^{Nd}$. If we ignore the spin and statistics, the state of the particles is described by the unit ray of the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^{Nd}) = \left\{ u(\underline{x}) \colon \|u\|^2 = \int_{\mathbb{R}^{Nd}} |u(x_1, \dots, x_N)|^2 dx_1 \dots dx_N < \infty \right\}.$$

We consider the situation that the particles are placed in the (classical) electro-magnetic field described by the electric scalar and the magnetic vector potentials given respectively by $\varphi(t,x)$ and $A(t,x) = (A_1(t,x), \dots, A_d(t,x))$, $(t,x) \in \mathbb{R} \times$

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$\mathbb{R}^d$ and they are interacting each other via the inter-particle forces given by the potential $V(t, \underline{x})$. If single-particle forces from additional external sources are present, we include them into $V(t, \underline{x})$. Then, the Hamiltonian of the system is given by

$$H(t) = \sum_{j=1}^{N} \left(\frac{1}{2m_j}(-i\hbar\nabla_j - e_j A(t, x_j))^2 + e_j\varphi(t, x_j) \right) + V(t, \underline{x}), \tag{1.1}$$

where $\nabla_j = (\partial/\partial x_{j1}, \dots, \partial/\partial x_{jd})$ for $1 \le j \le N$, $\hbar = h/2\pi$ and h is the Planck constant and, the dynamics is governed by the Schrödinger equation

$$i\hbar\frac{du}{dt} = H(t)u(t) \tag{1.2}$$

for $\mathcal{H}$-valued function $u(t) = u(t, \underline{x})$ of $t \in \mathbb{R}$. Hereafter we set $\hbar = 1$.

In this paper, we prove under rather general assumptions on the potentials which will be made precise below that Eqn. (1.2) generates a unique dynamics of the particles, or it uniquely generates a strongly continuous family of unitary operators $\{U(t, s) \colon -\infty < t, s < \infty\}$ on $\mathcal{H}$ such that $u(t) = U(t, s)f$ for $f \in \mathcal{H}$ produces the solution of (1.2) which satisfies the initial condition $u(s) = f$. We call $\{U(t, s)\}$ the *unitary propagator* for (1.2). It satisfies the Chapman-Kolmogorov equation

$$bU(t, s) = U(t, r)U(r, s), \quad U(t, t) = \mathbf{1}, \quad t, s, r \in \mathbb{R}, \tag{1.3}$$

where $\mathbf{1}$ is the identity operator on $\mathcal{H}$; we also prove, under slightly stronger assumptions, that $\{U(t, s)\}$ possesses the domain of definition $\Sigma(2)$ of the quantum harmonic oscillator as an invariant subspace such that $u(t) = U(t, s)f$ with $f \in \Sigma(2)$ is C^1 function of t with values in $\mathcal{H}$ and it satisfies (1.2) as an evolution equation in $\mathcal{H}$.

This is an improvement and an extension to N-particle systems of author's previous papers [23, 24] on the same subject for the case $N = 1$ and, before stating the main theorems, we think it appropriate to briefly touch upon the history of the subject.

The existence and the uniqueness of the unitary propagator for Schrödinger equations is certainly one of the most fundamental and the oldest problems in mathematics for quantum mechanics and it has been intensively and deeply studied by many authors since its advent (cf. [6]). If the Hamiltonian $H(t) = H$ is independent of time, the problem is virtually equivalent to the selfadjointness of H and, after a long and extensive study by various authors since Kato's seminal paper [12], it is now considered that the problem has almost been settled (see e.g. [5], [20] and reference therein for a large and rich literature).

For time dependent Hamiltonians $H(t)$, many and various methods have likewise been invented by many authors for producing the unitary propagator for (1.2). Adaptations of energy methods for the Cauchy problem for hyperbolic equations (e.g. [18, 22, 10]), the method of parabolic regularization ([18]) and the application of the theory of temporally inhomogeneous semi-groups (e.g.

[26, 20, 14, 15]), which we simply call semigroups in the sequel, to mention a few.

Among these, we think that the application of semigroup theory is the simplest and the furthest reaching, particularly for multi-particle Schrödinger equations and, most authors refer to this method for the existence of propagators. This theory, however, requires conditions like $D(H(t))$ *is t-independent and* $\partial_t H(t)$ *is* $H(t)$*- or* $H(t)$*-form bounded* or similar ones when $D(H(t))$ is t-dependent, which often impose rather strong restrictions on potentials, see e.g. [25, 2] where the existence of a unique unitary propagator for the case $N = 1$ is proved when potentials satisfy conditions which are almost necessary for $H(t)$ to be selfadjoint but under rather strong assumptions on the time derivative.

To see that the lack of this property can lead to the breakdown of the uniqueness of the propagator, we consider the following example:

$$i\partial_t u = \frac{1}{2}(-i\nabla + \sigma t x\langle x\rangle^{\sigma-2})^2 u + C\langle x\rangle^\sigma u = H_C(t)u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \tag{1.4}$$

where $\sigma \geq 0$ and $C > 0$. The operator $H_C(t)$ is seladjoint on $\mathcal{H}$ with maximal domain and is unitarily equivalent via $T(t)u(x) = e^{it\langle x\rangle^\sigma}u(x)$ to

$$T(t)^* H_C(t) T(t) = -\frac{1}{2}\Delta + C\langle x\rangle^\sigma \equiv H_{C,0}. \tag{1.5}$$

Thus, $\partial_t H_C(t) = -\frac{i\sigma}{2}T(t)\left(x\langle x\rangle^{\sigma-2}\cdot\nabla + \nabla\cdot x\langle x\rangle^{\sigma-2}\right)T(t)^*$ is *not* bounded with respect to $H_C(t)$ if $\sigma > 2$ for any $C > 0$ (but it is if $\sigma \leq 2$). On the other hand, the change of gauge $v(t,x) = T(t)u(t,x)$ transforms (1.4) into

$$i\partial_t v(t) = \left(-\frac{1}{2}\Delta + (C-1)\langle x\rangle^\sigma\right)v(t) = H_{C-1,0}v(t) \tag{1.6}$$

and, as is well known ([20]), $H_{C-1,0}$ is not essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$, if $\sigma > 2$ and $C < 1$, and has an infinitely number of selfadjoint extensions $\{H_\lambda\colon \lambda \in \Lambda\}$ each of which generates different dynamics for the same equation (1.4), $U_\lambda(t,s) = T(t)^* e^{-i(t-s)H_\lambda}T(s)$, breaking the uniqueness. Note, however, that $H_{C-1,0}|_{C_0^\infty(\mathbb{R}^d)}$ is essentially selfadjoint if $C \geq 1$ and it generates a unique dymanics for (1.4). Incidentally, $H_C(t)$ with $\sigma > 2$ and $C > 0$ has purely discrete spectrum with super-exponentially decreasing eigenfunctions by virtue of (1.5) and, this shows that the similarity of the appearance or the spectral properties of the Hamiltonian does not guarantees the same for the dynamics. Notice also very different dynamics of the corresponding classical mechanical particles for $C < 1$ and $C \geq 1$.

In the example above, the break down of uniqueness happens only when $\sigma > 2$ and the situation is very different if $\sigma \leq 2$. This is true in general and we have shown in [23] and [24] that, for the case $N = 1$, if $A(t,x)$ and $\varphi(t,x)$ are smooth and grow linearly or quadratically as $|x| \to \infty$ respectively then, Eqn. (1.2) generates a unitary propagator uniquely with the invariant subspace $\Sigma(2)$ when $V(t,x)$ is locally and spatially critically singular for the

selfadjointness of $H(t)$ and spatial singularities of $\partial_t V(t,x)$ can be stronger than those of $V(t,x)$ itself. For example, it is proven that, when centers of forces $y_1(t),\ldots,y_M(t) \in \mathbb{R}^3$ are moving smoothly,

$$i\partial_t u = \left(-\frac{1}{2m}\Delta + \sum_{l=1}^{M} \frac{Z_l}{|x_j - y_l(t)|^\gamma}\right) u \tag{1.7}$$

generates a unique dynamics if $\gamma < 3/2$ while $\partial_t |x - y_l(t)|^{-\gamma}$ are $-\Delta$-form bounded only when $\gamma \le 1$ (see [8] for the result for N-body Coulomb system).

Then, it is the purpose of this paper that we improve and extend the results of [23, 24] to N-particle systems, viz. by restricting the behavior of $A(t,x)$ and $\varphi(t,x)$ as $|x| \to \infty$ as above, we build a theory which guarantees the existence and the uniqueness of unitary propagators and which is general enough to cover most of conceivable applications in physics. We simultaneously show under a mild additional condition that the propagators thus obtained possess $\Sigma(2)$ as an invariant subspace with the properties mentioned above.

We now state main results of this paper precisely. For a function $f(t,x,\ldots)$ of $(t,x,\ldots)$ and $l = 0,1,\ldots,\infty$, we write $f \in C_x^l$, $f \in C_{(t,x)}^l$ and etc. if f is of class C^l with respect to variables x, (t,x) and etc.respectively. The skew symmetric $d \times d$ matrix

$$B(t,x) = (B_{jk}(t,x)) = (\partial A_k/\partial x_j - \partial A_j/\partial x_k), \quad j,k = 1,\ldots,d \tag{1.8}$$

is the magnetic field generated by $A(t,x)$. Here in (1.8), $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$ and x_j is not the position of j-th particle. We apologize for this double use of the notation and hope this causes no confusions. For a vector $a \in \mathbb{R}^n$ and an $n \times m$ matrix C, $|a|$ and $|C|$ are respectively the Euclidean length of a and the norm of C as a linear operator from $\mathbb{C}^m$ to $\mathbb{C}^n$ and $\langle a \rangle = (1+|a|^2)^{1/2}$. For multi-index $\alpha = (\alpha_1,\ldots,\alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Assumption 1.1. *Potentials $\varphi(t,x)$ and $A(t,x)$ are real valued, $\varphi, A \in C_x^\infty$ and, for any multi-index α, $\partial_x^\alpha \varphi \in C_{(t,x)}^0$ and $\partial_x^\alpha A \in C_{(t,x)}^1$. Moreover, the followings are satisfied for compact intervals $I \subset \mathbb{R}$:*

(1) *For any α with $|\alpha| \ge 2$, there exists a constant C_α such that*

$$|\partial_x^\alpha \varphi(t,x)| \le C_\alpha, \quad (t,x) \in I \times \mathbb{R}^d. \tag{1.9}$$

(2) *For any α with $|\alpha| \ge 1$, there exist $\varepsilon_\alpha > 0$ and C_α such that*

$$|\partial_x^\alpha B(t,x)| \le C_\alpha \langle x \rangle^{-1-\varepsilon_\alpha}, \tag{1.10}$$

$$|\partial_x^\alpha A(t,x)| + |\partial_x^\alpha \partial_t A(t,x)| \le C_\alpha, \quad (t,x) \in I \times \mathbb{R}^d. \tag{1.11}$$

We remark that (1.10) implies $\lim_{|x|\to\infty} B(t,x) = B(t)$ exists uniformly for $t \in I$ and $|B(t,x) - B(t)| \le C\langle x \rangle^{-\varepsilon}$, $\varepsilon > 0$. Thus, $B(t,x)$ is spatially a long range perturbation of a constant magnetic field $B(t)$.

We assume that $V(t, \underline{x})$ is the sum of potentials $V_D(t, \underline{x}_{D,r})$ of $|D|$-body interactions among particles in $D \subset (1, \dots, N)$ for the case $|D| \geq 2$ and potentials $V_j(t, x_j)$ of single-body external forces acting on the j-th particle for the case $D = \{j\}$, $1 \leq j \leq N$ (we may also consider that $V_j(t, x_j)$ is the non-smooth part of the electric scalar potential):

$$V(t, \underline{x}) = \sum_{D \subset (1, \dots, N)} V_D(t, \underline{x}_{D,r}). \tag{1.12}$$

Here, if $|D| \geq 2$, $V_D(t, \underline{x}_{D,r})$ are functions of $\underline{x}_{D,r}$, the positions of particles in D relative to the center of mass $x_{D,c}$ of D:

$$x_{D,c} = \sum_{j \in D} m_j x_j / \sum_{i \in D} m_j \tag{1.13}$$

and, we define $\underline{x}_{D,r} = x_j$ as a convention .

For stating the conditions on $V(t, \underline{x})$ precisely and also for later uses, we introduce some notation. We write $X = \mathbb{R}^{Nd}$ and define the inner product of $\underline{x} = (x_1, \dots, x_N)$ and $\underline{y} = (y_1, \dots, y_N) \in X$ by

$$(\underline{x}, \underline{y})_X = \sum m_j (x_j, y_j)_{\mathbb{R}^d}. \tag{1.14}$$

The configuration space of particles in $D = \{j_1, \dots, j_n\} \subset \{1, \dots, N\}$ is

$$X_D = \{\underline{x}_D = (x_{j_1}, \dots, x_{j_n})\colon x_{j_k} \in \mathbb{R}^d,\ k = 1, \dots, n\} = \mathbb{R}^{nd}$$

which is considered in the natural way as a subspace of X. The configuration space of the center of mass of D is defined by

$$X_{D,c} = \{\underline{x}_D = (x_{j_1}, \dots, x_{j_n}) \in X_D \colon x_{j_1} = \dots = x_{j_n}\} \simeq \mathbb{R}^d.$$

The projection of $\underline{x}_D = (x_{j_1}, \dots, x_{j_n})$ to $X_{D,c}$ is given by $\underline{x}_{D,c} = (x_{D,c}, \dots, x_{D,c})$. The configuration space of the motion of particles in D relative to $x_{D,c}$ is the orthogonal complement of $X_{D,c}$ within X_D:

$$X_{D,r} = X_D \ominus X_{D,c} \simeq \mathbb{R}^{d(n-1)}, \quad \underline{x}_D - \underline{x}_{D,c} = (r_{j_1}, \dots, r_{j_n}). \tag{1.15}$$

We take $x_{D,c}$ as the coordinates of $X_{D,c}$ and choose coordinates $x_{D,r}$ of $X_{D,r}$ (e.g. $x_{D,r} = x_2 - x_1$ if $D = \{1, 2\}$) such that

$$dx_{D,c} dx_{D,r} = dx_D. \tag{1.16}$$

If $D = \{j\}$, we define $X_{D,r} = \mathbb{R}^d$, $X_{D,c} = \{0\}$ and $\underline{x}_{D,r} = x_j$ as a convention.

$$n_D = \dim X_{D,r} = (|D| - 1)d, \text{ if } |D| \geq 2; \quad n_D = d, \text{ if } |D| = 1.$$

Recall that for Banach spaces $X_1, \dots, X_n$ which are subspaces of a linear topological space Y, the sum space $\Sigma = \sum X_j$ and intersection space $\Delta = \cap X_j$ are Banach spaces with the respective norms

$$\|u\|_\Sigma = \inf\{\sum \|u_j\| \colon u = u_1 + \dots + u_n\}, \quad \|u\|_\Delta = \|u\|_{X_1} + \dots + \|u\|_{X_n}.$$

Definition 1.2. *For $D \subset \{1,\dots,N\}$, $1 \le a,p \le \infty$ and compact intervals $I \subset \mathbb{R}$, we define three Banach spaces $\mathcal{I}_D^{a,p}(I)$ and $\tilde{\mathcal{I}}_D^{\infty,p}(I)$ by*

$$\mathcal{I}_D^{a,p}(I) = L^a(I, L^p(X_{D,r})) + L^1(I, L^\infty(X_{D,r})). \tag{1.17}$$

$$\tilde{\mathcal{I}}_D^{\infty,p}(I) = C(I, L^p(X_{D,r})) + L^1(I, L^\infty(X_{D,r})). \tag{1.18}$$

$$\mathcal{I}_D^{cont,p}(I) = C(I, L^p(X_{D,r})) + C(I, L^\infty(X_{D,r})). \tag{1.19}$$

For functions $V_D(t,\underline{x}_D)$ of $(t,\underline{x}_D) \in \mathbb{R}\times X_D$, we say $V_D \in \mathcal{I}_{\mathrm{loc},D}^{a,p}$ (resp. $V_D \in \tilde{\mathcal{I}}_{\mathrm{loc},D}^{\infty,p}$) if $V_D \in \mathcal{I}_D^{a,p}(I)$ (resp. $V_D \in \tilde{\mathcal{I}}_{\mathrm{loc},D}^{\infty,p}(I)$) for compact intervals I. Abusing notation, we write $\mathcal{I}_D^{cont,p}$ for $\mathcal{I}_D^{cont,p}(\mathbb{R})$.

We have $\mathcal{I}_D^{a,p}(I) \subset \mathcal{I}_D^{\tilde{a},\tilde{p}}(I)$ if $\tilde{a} \le a$ and $\tilde{p} \le p$ and $\mathcal{I}_D^{cont,p}(I) \subset \tilde{\mathcal{I}}_D^{\infty,p}(I) \subset \mathcal{I}_D^{a,p}(I)$ for any $1 \le a \le \infty$. We define $a(p_D)$ by

$$\frac{1}{a(p_D)} = 1 - \frac{n_D}{2p_D}, \quad \text{for} \quad \frac{n_D}{2} < p_D \le \infty. \tag{1.20}$$

Assumption 1.3. *$V(t,\underline{x})$ is given by* (1.12) *with $V_D(t,\underline{x}_{D,r})$ which satisfies either $V_D \in \tilde{\mathcal{I}}_{\mathrm{loc},D}^{\infty,n_D/2}$ or $V_D \in \mathcal{I}_{\mathrm{loc},D}^{a(p_D),p_D}$ for some $n_D/2 < p_D \le \infty$.*

As $a(p_D)$ decreases with p_D, $V_D \in \mathcal{I}_{\mathrm{loc},D}^{a(p_D),p_D}$ is the smoother in t if it is locally the more singular in $\underline{x}_{D,r}$. V_D is not necessary $-\Delta_{\underline{x}_{D,r}}$-bounded when $n_D \le 4$.

We define $D^c = \{1,\dots,N\} \setminus D$ and $\mathcal{H}_D = L^2(X_{D,c}) \otimes L^2(X_{D^c})$. We have

$$\mathcal{H} = L^2(X_{D,r}) \otimes \mathcal{H}_D = L^2(X_{D,r}, \mathcal{H}_D), \quad D \subset \{1,\dots,N\}.$$

Using the index p_D of Assumption 1.3 for $V_D(t,\underline{x}_{D,r})$, we define l_D and θ_D by

$$\frac{1}{l_D} = \frac{1}{2} - \frac{1}{2p_D}, \quad \frac{1}{\theta_D} = \frac{n_D}{4p_D}. \tag{1.21}$$

and, for intervals I, define Banach space $\mathcal{X}(I)$ of functions of $(t,\underline{x}) \in I \times X$ by

$$\mathcal{X}(I) = \cap_{D\subset\{1,\dots,N\}} L^{\theta_D}(I, L^{l_D}(X_{D,r}, \mathcal{H}_D)) \cap C(I,\mathcal{H}), \tag{1.22}$$

$$\|u\|_{\mathcal{X}(I)} = \sum_D \|u\|_{L^{\theta_D}(I, L^{l_D}(X_{D,r},\mathcal{H}_D))} + \|u\|_{C(I,\mathcal{H})}. \tag{1.23}$$

For $k = 0,1,\dots$, we write $\Sigma(k)$ for $\Sigma(k) = \{u\colon x^\alpha\partial^\beta u \in L^2(\mathbb{R}^n),\ |\alpha+\beta| \le k\}$ indiscriminately of the dimension of the space $\mathbb{R}^n$. $\Sigma(k)$ is the Hilbert space with the norm $\|u\|_{\Sigma(k)}$:

$$\|u\|_{\Sigma(k)}^2 = \sum \left\{ \|x^\alpha\partial^\beta u\|_{L^2}^2 \colon |\alpha+\beta| \le k \right\}.$$

$\Sigma(-k)$ is the dual space of $\Sigma(k)$ with respect to the inner product of $L^2(\mathbb{R}^n)$.

Theorem 1.4. *Suppose that (φ, A) and V satisfy* Assumptions 1.1 and 1.3 *respectively. Then, there uniquely exists a unitary propagator $\{U(t,s)\colon t,s \in \mathbb{R}\}$ for Eqn.* (1.2) *such that, for any $s \in \mathbb{R}$ and $f \in \mathcal{H}$, $u(t) = U(t,s)f$ satisfies the following properties:*

(1) *For compact intervals $I \subset \mathbb{R}$, $u \in \mathcal{X}(I)$ and for a constant C,*

$$\|u\|_{\mathcal{X}(I)} \le C\|f\|_{\mathcal{H}}, \quad f \in \mathcal{H}. \tag{1.24}$$

(2) *The function $u(t)$ is a locally absolutely continuous (AC for short) function of $t \in \mathbb{R}$ with values in $\Sigma(-2)$ and it satisfies in $\Sigma(-2)$ the equation*

$$i\frac{du}{dt} = H(t)u, \quad \text{a.e.}\, t \in \mathbb{R}. \tag{1.25}$$

Remark 1.5. The pair of indices (λ, σ) is called D-admissible Strichartz pair if it satisfies

$$0 \le \frac{2}{\sigma} = n_D\left(\frac{1}{2} - \frac{1}{\lambda}\right) \le 1. \tag{1.26}$$

Hence, (l_D, θ_D) of (1.21) is a D-admissible pair. Interpolating (1.24) with with the unitary property of the propagator $\|u\|_{L^\infty(I,L^2(X_{D,r},\mathcal{H}_D))} = \|f\|_{\mathcal{H}}$, we see that $u(t) = U(t,s)f$ satisfies the Strichartz inequality for all D-admissible Strichartz pairs (λ, σ) and (μ, τ) such that $2 \le \lambda, \mu \le l_D$:

$$\sup_{s\in I} \|U(t,s)f\|_{L^\sigma(I,L^{\lambda,2}_{D_1})} \le C_I\|f\|_{\mathcal{H}} \tag{1.27}$$

and, hence, the two others:

$$\sup_{s\in I}\left\|\int_I U(s,t)u(t)dt\right\|_{\mathcal{H}} \le C_I\|u\|_{L^{\sigma'}(I,L^{\lambda',2}_{D_1})}, \tag{1.28}$$

$$\left\|\int_s^t U(t,r)u(r)dr\right\|_{L^\sigma([s,s+L],L^{\lambda,2}_{D_1})} \le C_L\|u\|_{L^{\tau'}([s,s+L],L^{\mu',2}_{D_2})}, \tag{1.29}$$

where σ', λ' and etc. are dual exponents of σ, λ and etc: $1/\sigma + 1/\sigma' = 1$ and etc.(we refer to the proof of Lemma 2.3 to see how (1.28) and (1.29) follow from (1.27)). Here we recall that $l_D = 2p_D/(p_D - 1)$, which implies the smaller set of D-admissible pairs for larger p_D. This seemingly contradicting phenomenon happens because we have chosen the smallest possible $a(p_D)$ for a given p_D such that $V_D \in \mathcal{I}_D^{a,p_D}$ for obtaing the most general statement of the theorem. Thus, if $V_D \in \mathcal{I}_D^{a,p_D}$ is satisfied for $a > a(p_D)$, then l_D can be replaced by the larger $2q_D/(q_D - 1)$ such that $a = a(q_D)$. In particular, if $V_D \in \tilde{\mathcal{I}}_D^{\infty,p}$ for $p \ge n_D/2$, then, Strichartz estimates are satisfied for the full range of λ: $2 \le \lambda \le 2n_D/(n_D - 2)$.

For $n_D/2 \le p_D \le \infty$ of Theorem 1.4, we define

$$\tilde{p}_D = \max(2, p_D), \quad b_D = \frac{4p_D}{4p_D - n_D}, \tag{1.30}$$

$$q_D = \frac{2n_D p_D}{n_D + 4p_D} \text{ if } n_D \ge 4, \quad q_D = \frac{2p_D}{p_D + 1} \text{ if } n_D = 3. \tag{1.31}$$

Assumption 1.6. *$V(t,\underline{x})$ is given by* (1.12) *with $V_D(t,\underline{x}_D)$ such that $V_D \in \mathcal{I}_D^{cont,\tilde{p}_D}$ and $\partial_t V_D \in \mathcal{I}_{D,\mathrm{loc}}^{b_D,q_D}$ for some $n_D/2 \le p_D \le \infty$.*

Remark 1.7. Definition (1.31) may be written as

$$\frac{1}{q_D} = \frac{1}{p_D} + \left(\frac{2}{n_D} - \frac{1}{2p_D}\right),\ n_D \ge 4; \quad \frac{1}{q_D} = \frac{1}{\tilde{p}_D} + \left(\frac{1}{2} + \frac{1}{2p_D} - \frac{1}{\tilde{p}_D}\right),\ n_D = 3.$$

Thus, as for the singularities of the type $|x|^{-a}$, $\partial_t V_D(t, x_{D,r})$ can be more singular than $V_D(t,x_{D,r})$ by $C|x_{D,r}|^{-\left(2-\frac{n_D}{2p_D}\right)+\varepsilon}$, $\varepsilon > 0$ if $n_D \ge 4$ and, if $n_D = 3$, by $C|x_{D,r}|^{-3\left(\frac{1}{2}-\frac{1}{2p_D}\right)+\varepsilon}$ if $p_D \ge 2$ and $C|x_{D,r}|^{-\frac{3}{2p_D}+\varepsilon}$ if $3/2 \le p_D \le 2$. $\varepsilon > 0$. Thus, the exponents are $-1+\varepsilon$ for all $n_D \ge 3$ if $p_D = n_D/2$ whereas for large p_D they are close to -2 if $n_D \ge 4$ and to $-3/2$ if $n_D = 3$.

Theorem 1.8. *Suppose that (φ, A) and $V(t,\underline{x})$ satisfy* Assumptions 1.1 and 1.6 *respectively. Suppose, in addition, $\varphi \in C^1_{(t,x)}$ and*

$$|A(t,x)| + |\partial_t A(t,x)| \le C\langle x\rangle, \quad |\partial_t \varphi(t,x)| \le C\langle x\rangle^2, \quad (t,x) \in I \times \mathbb{R}^d \qquad (1.32)$$

for compact intervals I. Then, $U(t,s)$ of Theorem 1.4 *satisfies the following: If $f \in \Sigma(2)$, then $u(t) = U(t,s)f \in C(\mathbb{R}, \Sigma(2)) \cap C^1(\mathbb{R}, \mathcal{H})$ and $\partial_t u \in \mathcal{X}(I)$ for any compact interval I. It satisfies Eqn.* (1.2) *as an evolution equation in $\mathcal{H}$.*

Remark 1.9. The proof of Theorem 1.8 actually shows that, for $f \in \Sigma(2)$, the solution $u(t)$ satisfies $\partial_t u \in \mathcal{X}(I)$ for compact intervals I. Since $Vu \in \mathcal{X}(I)$ if $u \in C(I, \Sigma(2))$ and V satisfies Assumption 1.6, we have $H_0(t)u \in \mathcal{X}(I)$ as well. Here $H_0(t)$ is the Hamiltonian for N independent particles in the field:

$$H_0(t) = \sum_{j=1}^{N} \left(\frac{1}{2m_j}(-i\hbar\nabla_j - e_j A(t,x_j))^2 + e_j\varphi(t,x_j)\right). \qquad (1.33)$$

We describe here the plan of the paper. In Sec. 2 we record some results which will be used in later sections: We first recall some results of [9, 24] on fundamental solutions, i.e. integral kernels of the unitary propagators of single particle Schrödinger equations and, prove Strichartz estimates of new type which is tailored for our purpose for the propagator associated with the Hamiltonian $H_0(t)$. With this new Strichartz estimates, we prove Theorem 1.4 in Sec. 3. The argument is based on the contraction mapping principle and is a straightforward extension of that in [23, 24]. We prove Theorem 1.8 in Sec. 4 after a few preparations. In subsec. 4.1, we apply gauge transformation to Eqn. (1.2) and reduce the poof to the case when $\varphi(t,x) \ge C\langle x\rangle^2$ with a sufficiently large constant $C > 0$. Under this condition, we prove that $H_0(t)$ with domain $D(H_0(t)) = \Sigma(2)$ is selfadjoint and $H_0(t) \ge 1$ for all $t \in \mathbb{R}$. We complete the proof of Theorem 1.8 in subsec. 4.2.

Most of the notation is standard. For Banach spaces X and Y, $\mathbf{B}(X,Y)$ is the Banach space of all bounded operators from X to Y and $\mathbf{B}(X) = \mathbf{B}(X,X)$. The

Laplacian in Euclidean space of various dimensions is denoted indiscriminately by Δ. If $K(x,y)$ is a function of $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$, then $\frac{\partial^2 K}{\partial x \partial y}$ is the $d \times d$-matrix with (j,k) elements $\frac{\partial^2 K}{\partial x_j \partial y_k}$, $1 \le j,k \le d$. For $n = 0, 1, \ldots$, $O(\langle \underline{x} \rangle^n)$ and etc. is a function which is bounded by $C\langle \underline{x} \rangle^n$ and etc. Various constants are denoted by the same letter C when their specific values are not important and the same C may differ from one place to the other. *In what follows in this paper, we arbitrarily take and fix a (large) compact interval I_0 and assume that time variables and intervals are always inside I_0.*

2 Preliminaries

In this section we record several known facts which we use for proving Theorems.

2.1 Fundamental solutions

We use the following well known theorem on single particle Schrödinger equations ([9, 24]):

Theorem 2.1. *Suppose that A and φ satisfy* Assumption 1.1*, $m > 0$ and $e \in \mathbb{R}$. Then, there exists a unique unitary propagator $\{U_{sing}(t,s) \colon t, s \in \mathbb{R}\}$ on $L^2(\mathbb{R}^d)$ for the Schrödinger equation*

$$i\partial_t u(t,x) = \left(\frac{1}{2m}(-i\nabla - eA(t,x))^2 + e\varphi(t,x) \right) u(t,x). \tag{2.1}$$

The propagator satisfies the following properties:

(1) *For $f \in \Sigma(k)$, $u(t) = U_{sing}(t,s)f$ satisfies $u \in C(\mathbb{R}, \Sigma(k)) \cap C^1(\mathbb{R}, \Sigma(k-2))$, $k = 0, 1, \ldots$. In particular, $U_{sing}(t,s)$ is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$.*

(2) *There exists $T > 0$ such that, for $0 < |t-s| < T$, $U_{sing}(t,s)$ is an oscillatory integral operator (OIOp for short) of the form*

$$U_{sing}(t,s)f(x) = \frac{m^{d/2}}{(2\pi i(t-s))^{d/2}} \int_{\mathbb{R}^d} e^{iS(t,s,x,y)} b(t,s,x,y) f(y) dy. \tag{2.2}$$

Here $S(t,s,x,y)$ and $b(t,s,x,y)$ satisfy the following properties:

(a) *For any α, β, $S \in C^\infty_{(x,y)}$ and $\partial_x^\alpha \partial_y^\beta S \in C^1_{(t,s,x,y)}$. For $|\alpha| + |\beta| \ge 2$, there exists a constant $C_{\alpha\beta}$ such that*

$$\left| \partial_x^\alpha \partial_y^\beta \left(S(t,s,x,y) - \frac{m(x-y)^2}{2(t-s)} \right) \right| \le C_{\alpha\beta}, \quad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{2.3}$$

(b) *For any α, β, $b \in C^\infty_{(x,y)}$ and $\partial_x^\alpha \partial_y^\beta b \in C^1_{(t,s,x,y)}$. There exists a constant $C_{\alpha\beta}$ such that*

$$|\partial_x^\alpha \partial_y^\beta (b(t,s,x,y) - 1)| \le C_{\alpha\beta} |t-s|, \quad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{2.4}$$

Incidentally $S(t,s,x,y)$ is the action integral of the classical path $(p(\tau), q(\tau))$ corresponding to (2.1) such that $q(s) = y$ and $q(t) = x$, viz.

$$\frac{dq}{d\tau} = \frac{\partial h}{\partial p}, \quad \frac{dp}{d\tau} = -\frac{\partial h}{\partial p}, \quad h(\tau, p, q) = \frac{1}{2m}(p - eA(\tau, q))^2 + e\varphi(\tau, q).$$

When $V = 0$, $H(t) = H_0(t)$ and (1.2) becomes

$$i\partial_t u = (H_{0,1}(t) + \cdots + H_{0,N}(t))u, \tag{2.5}$$

$$H_{0,j}(t) = \frac{1}{2m_j}(-i\nabla_j - e_j A(t, x_j))^2 + e_j \varphi(t, x_j), \quad j = 1, \ldots, N. \tag{2.6}$$

The unitary propagator for (2.5) is given by the tensor product:

$$U_0(t,s) = U_{0,1}(t,s) \otimes \cdots \otimes U_{0,N}(t,s) \text{ on } \mathcal{H} = \otimes_{j=1}^N L^2(\mathbb{R}^d) \tag{2.7}$$

of the propagators $U_{0,j}(t,s)$ for the j-th particle:

$$i\partial_t u = H_{0,j}(t)u, \quad j = 1, \ldots, N. \tag{2.8}$$

By virtue of Theorem 2.1, $\{U_0(t,s) \colon -\infty < t, s < \infty\}$ is strongly continuous in $\Sigma(k)$ and C^1 from $\Sigma(k)$ to $\Sigma(k-2)$ for any $k = 0, 1, \ldots$ and, it is strongly C^1 in $\mathcal{S}(\mathbb{R}^{Nd})$. There exists $T > 0$ such that for $0 < |t-s| < T$ all $U_{0,j}(t,s)$ are OIOp's of the form

$$U_{0,j}(t,s)f(x) = \frac{m_j^{d/2}}{(2\pi i(t-s))^{d/2}} \int_{\mathbb{R}^d} e^{iS_j(t,s,x,y)} b_j(t,s,x,y) f(y) dy, \tag{2.9}$$

with $S_j(t,s,x,y)$ and $b_j(t,s,x,y)$ which satisfy the properties corresponding to (a) and (b) of Theorem 2.1. *We take this $T > 0$ and, in what follows, we will make it further smaller when it becomes necessary.*

For $D = \{j_1, \ldots, j_n\} \subset \{1, \ldots, N\}$, $U_{0,D}(t,s)$ is the unitary propagator on $L^2(X_D)$ for n independent particles inside D:

$$U_{0,D}(t,s) = U_{0,j_1}(t,s) \otimes \cdots \otimes U_{0,j_n}(t,s). \tag{2.10}$$

We often consider $U_{0,D}(t,s)$ as an operator on $L^2(X) = L^2(X_D) \otimes L^2(X_{D^c})$ by identifying it with $U_{0,D}(t,s) \otimes \mathbf{1}_{L^2(X_{D^c})}$, where $\mathbf{1}_{L^2(X_{D^c})}$ is the identity operator of $L^2(X_{D^c})$.

2.2 Strichartz estimates

From the decomposition $X = X_{D^c} \oplus X_{D,c} \oplus X_{D,r}$, we have $\mathcal{H} = L^2(X_{D,r}, \mathcal{H}_D)$, $\mathcal{H}_D = L^2(X_{D^c} \oplus X_{D,c})$, see (1.16). Then, for $1 \le p, q \le \infty$, we define

$$L_D^{p;q}(X) = L^p(X_{D,r}, L^q(X_{D^c} \oplus X_{D,c})), \tag{2.11}$$

$$\|u\|_{L_D^{p;q}} = \left(\int_{X_{D,r}} \|u(\underline{x}_{D,r}, \cdot)\|^p_{L^q(X_{D^c} \oplus X_{D,c})} d\underline{x}_{D,r} \right)^{1/p}. \tag{2.12}$$

Recall $n_D = \dim X_{D,r} = (|D|-1)d$ if $|D| \ge 2$ and $n_D = d$ if $|D| = 1$. The following lemma is the extension to N independent particles of the well-known L^p–L^q estimates for single particle Schrödinger equations (2.1).

Lemma 2.2. *Let $1 \le q \le 2 \le p \le \infty$ satisfy $1/p + 1/q = 1$. Then, for any $D \subset \{1, \dots, N\}$, there exists a constant C such that, for $0 < |t-s| \le T$,*

$$\|U_0(t,s)u\|_{L_D^{p,2}(X)} \le C|t-s|^{-n_D(1/2-1/p)}\|u\|_{L_D^{q,2}(X)}. \tag{2.13}$$

Proof. We prove the lemma when $D = \{1, \dots, N\}$, omitting subscript D from various notation, e.g. $X_c = X_{D,c}$, $X_r = X_{D,r}$ and $\underline{x} = (x_c, \underline{x}_r) \in X = X_c \oplus X_r$. The proof for other cases is similar. From (2.7) and (2.9)

$$\begin{aligned} U_0(t,s)u(x_c, \underline{x}_r) &= \frac{(m_1 \cdots m_N)^{d/2}}{(2\pi i(t-s))^{dN/2}} \\ &\quad \times \int_{\mathbb{R}^{Nd}} e^{i\sum S_j(t,s,x_j,y_j)} \prod b_j(t,s,x_j,y_j) u(y_c, \underline{y}_r) dy_c d\underline{y}_r. \end{aligned} \tag{2.14}$$

Using the notation (1.15) for $\underline{x} = \underline{x}_c + (r_1, \dots, r_N)$ and the corresponding for $\underline{y} = \underline{y}_c + (s_1, \dots, s_N)$, we write

$$F(t,s,\underline{x},\underline{y}) \equiv \sum_{j=1}^N S_j(t,s,x_j,y_j) = \sum_{j=1}^N S_j(t,s,x_c + r_j, y_c + s_j), \tag{2.15}$$

$$B(t,s,\underline{x},\underline{y}) \equiv \prod_{j=1}^N b_j(t,s,x_j,y_j) = \prod_{j=1}^N b_j(t,s,x_c + r_j, y_c + s_j). \tag{2.16}$$

Then, (2.3) and (2.4) respectively imply

$$\frac{\partial^2 F}{\partial x_c \partial y_c} = \sum_{j=1}^N \frac{\partial^2 S_j}{\partial x_c \partial y_c}(t,s,x_c + r_j, y_c + s_j) = \sum_{j=1}^N \frac{m_j}{t-s}\mathbf{1}_d + O(1), \tag{2.17}$$

$$|\partial_{x_c}^\alpha \partial_{y_c}^\beta B(t,s,\underline{x},\underline{y})| \le C_{\alpha\beta}, \quad |\alpha| + |\beta| \ge 0, \tag{2.18}$$

where $\mathbf{1}_d$ is the $d \times d$ unit matrix and $O(1)$ is $d \times d$ matrix whose components are functions bounded along with all derivatives with respect to the spatial variables $(\underline{x}, \underline{y}) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$. Then, the Minkowski inequality and the Asada-Fujiwara L^2-boundedness theorem for OIOp's ([1]) yield

$$\begin{aligned} &\frac{1}{(2\pi|t-s|)^{Nd/2}} \left\| \int_{X_r} \left(\int_{X_c} e^{iF(t,s,\underline{x},\underline{y})} B(t,s,\underline{x},\underline{y}) u(y_c, \underline{y}_r) dy_c \right) d\underline{y}_r \right\|_{L^2(X_c)} \\ &\quad \le \frac{1}{(2\pi|t-s|)^{Nd/2}} \int_{X_r} \left\| \int_{X_c} e^{iF(t,s,x,y)} B(t,s,x,y) u(y_c, \underline{y}_r) dy_c \right\|_{L^2(X_c)} d\underline{y}_r \\ &\quad \le \frac{C}{|t-s|^{d(N-1)/2}} \int_{X_r} \left\| u(\cdot, \underline{y}_r) \right\|_{L^2(X_c)} d\underline{y}_r. \end{aligned}$$

It follows that

$$\|U_0(t,s)u(x_c,\underline{x}_r)\|_{L_D^{\infty,2}} \le C|t-s|^{-d(N-1)/2}\|u(y_c,\underline{y}_r)\|_{L_D^{1,2}} \tag{2.19}$$

Thus, interpolating the inequality (2.19) with the unitary property of $U_0(t,s)$:

$$\|U_0(t,s)u\|_{L_D^{2,2}} = \|u\|_{L_D^{2,2}},$$

we obtain the desired result. □

Recall that the pair of indices (λ,σ) is called D-admissible if it satisfies (1.26). We have the following set of Strichartz' estimates for N independent particles in the external field (cf. Remark 1.5).

Lemma 2.3. *Let $D_1, D_2 \subset \{1,\dots,N\}$, (λ,σ) and (μ,τ) be admissible pairs for D_1 and D_2, respectively and λ' and etc. be dual exponents of λ and etc., viz. $1/\lambda + 1/\lambda' = 1$ and etc. Then, there exist a constant C such that the following estimates are satisfied for intervals $I \subset I_0$ and $[s,s+L] \subset I_0$:*

$$\|U_0(t,s)f\|_{L^\sigma(I,L_{D_1}^{\lambda,2})} \le C\|f\|_{\mathcal{H}}, \quad s \in \mathbb{R}. \tag{2.20}$$

$$\left\|\int_I U_0(s,t)u(t)dt\right\|_{\mathcal{H}} \le C\|u\|_{L^{\sigma'}(I,L_{D_1}^{\lambda',2})}, \quad s \in \mathbb{R}. \tag{2.21}$$

$$\left\|\int_s^t U_0(t,r)u(r)dr\right\|_{L^\sigma([s,s+L],L_{D_1}^{\lambda,2})} \le C\|u\|_{L^{\tau'}([s,s+L],L_{D_2}^{\mu',2})}, \quad s \in I. \tag{2.22}$$

The estimate (2.22) *likewise holds if $[s,s+L]$ is replaced by $[s-L,s] \subset I_0$.*

Proof. Define, for every fixed $s \in \mathbb{R}$,

$$U_{0,s}(t) = \begin{cases} U_0(t,s), & s-T<t<s+T, \\ 0, & T \le |t-s|. \end{cases}$$

Then, $\{U_{0,s}(t) \colon t \in \mathbb{R}\}$ satisfies

$$\text{either } U_{0,s}(t)U_{0,s}(r)^* = U_0(t,r) \text{ or } U_{0,s}(t)U_{0,s}(r)^* = 0$$

and Lemma 2.2 implies

$$\|U_{0,s}(t)f\|_{L_D^{2,2}} \le C\|f\|_{L_D^{2,2}}, \quad f \in L_D^{2,2}. \tag{2.23}$$

$$\|U_{0,s}(t)U_{0,s}(r)^*f\|_{L_D^{\infty,2}} \le C|t-r|^{-n_D/2}\|f\|_{L_D^{1,2}}, \quad f \in L_D^{1,2}. \tag{2.24}$$

Thus, if we consider $\{U_{0,s}(t) \colon t \in \mathbb{R}\}$ as the family of operators acting on functions of $\underline{x}_r \in X_{D,r}$ with values in the Hilbert space $\mathcal{H}_D = L^2(X_{D,c} \oplus X_{D,c})$, it satisfies the Keel-Tao conditions ([17]) for Strichartz estimates. Then, thanks to the fact that, for any Hilbert space $\mathcal{K}$,

$$L^\theta(\mathbb{R}^{d_1}, L^p(\mathbb{R}^{d_2},\mathcal{K}))^* = L^{\theta'}(\mathbb{R}^{d_1}, L^{p'}(\mathbb{R}^{d_2},\mathcal{K})) \tag{2.25}$$

when $1 \le \theta, \theta' < \infty$ and $1 < p, p' < \infty$ satisfy $1/\theta + 1/\theta' = 1$ and $1/p + 1/p' = 1$ (cf. e.g. [7], pp. 97–100), the proof of Strichartz estimates presented in [21] can be applied almost word by word to the vector valued functions and produces (2.20), (2.21) and (2.22) when $|I| < T$ and $L < T$ respectively.

We next prove (2.20), (2.21) and (2.22) when the time intervals $I = [T_1, T_2] \subset I_0$ and $[s, s+L] \subset I_0$ are of arbitrary size. Then we take a decomposition $t_0 = T_1 < t_1 < \cdots < t_n = T_2$ in such a way that $T/2 < t_j - t_{j-1} < T$ for $j = 1, \dots, n$. Then, the result (2.20) for $|I| < T$ implies

$$\begin{aligned}
\int_{T_1}^{T_2} \|U_0(t,s)f\|^\sigma_{L^{\lambda,2}_{D_1}} dt &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|U_0(t,s)f\|^\sigma_{L^{\lambda,2}_{D_1}} dt \\
&= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|U_{t_{j-1}}(t)U_0(t_{j-1},s)f\|^\sigma_{L^{\lambda,2}_{D_1}} dt \\
&\le C \sum_{j=1}^n \|U_0(t_{j-1},s)f\|^\sigma_{\mathcal{H}} \le Cn\|f\|^\sigma_{\mathcal{H}}
\end{aligned}$$

and (2.20) for $[T_1, T_2]$ follows. The estimate (2.21) follows from (2.20) by the well known duality argument. For proving (2.22), for shorting formulas, we write $F(t) = \int_s^t U_0(t,r)u(r)dr$. We again decompose $[s, s+L]$: $t_0 = s < t_1 < \cdots < t_n = s+L$ in such a way that $T/2 < t_j - t_{j-1} < T$ for $j = 1, \dots, n$. Then, denoting $\Delta_j = (t_{j-1}, t_j)$, we have

$$\begin{aligned}
&\|F\|_{L^\sigma([s,L+s],L^{\lambda,2}_{D_1})} \le \sum_{j=1}^n \|F\|_{L^\sigma(\Delta_j,L^{\lambda,2}_{D_1})} \\
&= \sum_{j=1}^n \left\| U_{t_{j-1}}(t) \int_s^{t_{j-1}} U_0(t_{j-1},r)u(r)dr + \int_{t_{j-1}}^t U_0(t,r)u(r)dr \right\|_{L^\sigma(\Delta_j,L^{\lambda,2}_{D_1})}.
\end{aligned}$$

We apply (2.20) and (2.21) to the first term in the sign of the norm and the short time result (2.22) to the second. Then, we see that the right hand side is bounded by

$$\begin{aligned}
&\sum_{j=1}^n \left(\left\| \int_s^{t_{j-1}} U_0(t_{j-1},r)u(r)dr \right\|_{\mathcal{H}} + \|u\|_{L^{\tau'}(\Delta_j,L^{\mu',2}_{D_2})} \right) \\
&\quad \le C \sum_{j=1}^n \left(\|u\|_{L^{\tau'}([s,t_{j-1}],L^{\mu',2}_{D_2})} + \|u\|_{L^{\tau'}(\Delta_j,L^{\mu',2}_{D_2})} \right) \\
&\quad \le C(n+1)^{1-1/\tau'} \|u\|_{L^{\tau'}([s,L+s],L^{\mu',2}_{D_2})}.
\end{aligned}$$

The proof of (2.22) with $[s-L, s]$ is similar. This completes the proof. □

We use the following version of Christ-Kiselev lemma [4] which appears in [21] and which is also used in the proof of Lemma 2.3.

Lemma 2.4. *Let Y and Z be Banach spaces and assume that $K(t,s)$ is a strongly measurable function of $t,s \in (a,b)$, $-\infty \le a < b \le \infty$ taking values in $\mathbf{B}(Y,Z)$ such that $\|K(t,s)\|_{\mathbf{B}(X,Y)}$ is locally integrable. Set*

$$Tf(t) = \int_a^b K(t,s)f(s)ds.$$

Assume that

$$\|Tf\|_{L^q((a,b),Z)} \le C_0\|f\|_{L^p((a,b),Y)}$$

Define

$$Wf(t) = \int_a^t K(t,s)f(s)ds$$

Then, if $1 \le p < q \le \infty$,

$$\|Wf\|_{L^q((a,b),Z)} \le C_0 C_{p,q}\|f\|_{L^p((a,b),Y)}$$

We often use the following estimates in the proof of Lemma 4.5

Lemma 2.5. *Let $I = [s, s+L]$ or $I = [s-L, s]$. Then, there exists a constant C such that*

$$\left\| \int_s^t U_0(t,r)u(r)dr \right\|_{\mathcal{X}(I)} \le C\|u\|_{L^1(I,\mathcal{H})}. \tag{2.26}$$

Proof. We prove the case $I = [s, s+L]$. The other case may be proves similarly. Let $M = s+L$. Then

$$\left\| \int_s^M U_0(s,r)u(r)dr \right\|_{\mathcal{H}} \le L\|u\|_{L^1(I,\mathcal{H})}$$

It follows by (2.20) that

$$\left\| \int_s^M U_0(t,r)u(r)dr \right\|_{\mathcal{X}(I)} \le C\|u\|_{L^1(I,\mathcal{H})}$$

Then, (2.26) follows by virtue of Lemma 2.4. ∎

3 Proof of Theorem 1.4

The proof of Theorem 1.4 is the adaptation of that of Theorem 1 of [24], using new function spaces $\mathcal{X}(I)$ defined above and new Strichartz estimates of Lemma 2.3.

Let $\mathcal{V}$ be the multiplication by $V(t,x)$:

$$(\mathcal{V}u)(t,\underline{x}) = V(t,\underline{x})u(t,\underline{x}) = \sum_{D\subset(1,\dots,N)} V_D(t,\underline{x}_{D,r})u(t,\underline{x})$$

and $\mathcal{G}_s$ be the integral operator defined by

$$(\mathcal{G}_s u)(t) = -i \int_s^t U_0(t,r)u(r)dr, \tag{3.1}$$

where $U_0(t,s)$ is the unitary propagator for $H_0(t)$ defined by (2.7). The Duhamel formula implies that (1.2) with the initial condition $u(s) = f \in \mathcal{H}$ is equivalent to the integral equation

$$u(t) = u_0(t) + (\mathcal{G}_s \mathcal{V} u)(t), \tag{3.2}$$

where we wrote $u_0(t) = U_0(t,s)f$. Define for intervals $I \subset I_0$ the function space $\mathcal{X}(I)$ by (1.22) by using indices l_D and θ_D of (1.21). As was remarked previously (l_D, θ_D) is a D-admissible pair. Along with $\mathcal{X}(I)$, we define another Banach space $\mathcal{X}^*(I)$ by

$$\mathcal{X}^*(I) = \sum_D L^{\theta'_D}(I, L_D^{l',2}) + L^1(I, \mathcal{H}),$$

$$\|u\|_{\mathcal{X}^*(I)} = \inf\{\sum_D \|u_D\|_{L^{\theta'_D}(I, L_D^{l'_D,2})} + \|u_1\|_{L^1(I,\mathcal{H})} \colon u = \sum_D u_D + u_1\},$$

where θ'_D and l'_D are dual exponents of θ_D and l_D respectively. The space $\mathcal{X}^*(I)$ is almost the dual space of $\mathcal{X}(I)$ but not exactly.

Lemma 3.1. *For a constant C independent of $I \subset I_0$ and $s \in I$ following statements are satisfied:*

(1) *For $f \in \mathcal{H}$, $U_0(t,s)f \in \mathcal{X}(I)$.*

(2) *The multiplication $\mathcal{V}$ is bounded from $\mathcal{X}(I)$ to $\mathcal{X}^*(I)$ and*

$$\|\mathcal{V}u\|_{\mathcal{X}^*(I)} \le C \max_D \|V_D\|_{\mathcal{I}_D^{a(p_D),p_D}(I)} \|u\|_{\mathcal{X}(I)}, \tag{3.3}$$

(3) *Integral operator $\mathcal{G}_s$ is bounded from $\mathcal{X}^*(I)$ to $\mathcal{X}(I)$ and*

$$\|\mathcal{G}_s u\|_{\mathcal{X}(I)} \le C \|u\|_{\mathcal{X}^*(I)}. \tag{3.4}$$

Proof. Write $a(p_D) = a_D$. Statement (1) is a result of (2.20) and (2.21). We have

$$\frac{1}{l_D} + \frac{1}{p_D} = \frac{1}{l'_D} \quad \text{and} \quad \frac{1}{\theta_D} + \frac{1}{a_D} = \frac{1}{\theta'_D}$$

by the definition (1.20) and (1.21). Thus, Hölder's inequality implies for $V_D = V_D^{(1)} + V_D^{(2)} \in L^{l_D}(I, L^{p_D}(X_{D,r}) + L^1(I, L^\infty(X_{D,r})$ that

$$\|V_D^{(1)}u\|_{L^{\theta'_D}(I, L_D^{l'_D,2})} \le \|V_D^{(1)}\|_{L^{a_D}(I, L^{p_D}(X_{D,r}))} \|u\|_{L^{\theta_D}(I, L_D^{l_D,2})},$$

$$\|V_D^{(2)}u\|_{L^1(I,\mathcal{H})} \le \|V_D^{(2)}\|_{L^1(I, L^\infty(X_{D,r}))} \|u\|_{L^\infty(I,\mathcal{H})}.$$

This implies (3.3) and statement (2) is proved. By virtue of (2.22), $\mathcal{G}_s$ is bounded from the sum space $\Sigma = \sum_D L^{\theta'_D}(I, L_D^{l'_D,2})$ to the intersection space $\cap_D L^{\theta_D}(I, L_D^{l_D,2})$. It is also bounded from Σ to $C(I,\mathcal{H})$, for $\mathcal{G}_s$ is bounded from Σ into $L^\infty(I,\mathcal{H})$, $\mathcal{G}_s u \in C(I,\mathcal{H})$ if $u \in C(I,\Sigma(2))$ and $C(I,\Sigma(2))$ is dense in Σ. By virtue of Minkowski's inequality and (2.20), $\mathcal{G}_s$ is bounded from $L^1(I,\mathcal{H})$ to $\mathcal{X}(I)$. Thus, statement (3) is satisfied. . $\square$

Proof of Theorem 1.4 Let $I \subset I_0$ be an interval. Then estimates (3.3) and (3.4) imply

$$\|\mathcal{G}_s \mathcal{V} u\|_{\mathcal{X}(I)} \le C \sum_D \|V_D\|_{\mathcal{I}_D^{a(p_D),p_D}(I)} \|u\|_{\mathcal{X}(I)},$$

where C is independent of I and $s \in I$. It is obvious that $\|V_D\|_{\mathcal{I}_D^{a(p_D),p_D}(I)} \to 0$ as $|I| \to 0$ if $a(p_D) < \infty$ or $p_D < n_D/2$. This is also true when $p_D = n_D/2$ and $V_D \in \tilde{\mathcal{I}}^{\infty,n_D/2}(I)$, for $f_M(t) \equiv \|V_D(t,x_{D,r})\chi_{\{|V_D(t,x_{D,r})|>M\}}\|_{L^p(X_{D,r})}$ is continuous, $\lim_{M\to\infty} f_M(t) = 0$ decreasingly, hence uniformly on I by Dini's theorem and $\|V_D(t,x_{D,r})\chi_{\{|V_D(t,x_{D,r})|\le M\}}\|_{L_D^{1,\infty}(I)} \le M|I| \to 0$ as $|I| \to 0$. Thus, $\mathcal{G}_s\mathcal{V}\colon \mathcal{X}(I) \to \mathcal{X}(I)$ is a contraction if I is sufficiently small, and (3.2) has a unique solution $u \in \mathcal{X}(I)$ for any $f \in \mathcal{H}$. It can be expressed as

$$u(t) = \Gamma_t (1-\mathcal{G}_s\mathcal{V})^{-1} T_s f, \quad (T_s f)(t) \equiv U_0(t,s) f,$$

where Γ_t is the evaluation operator at t, i.e. $\Gamma_t w = w(t)$ for $w \in \mathcal{X}(I)$. We define the operator $U(t,s)$ for $t,s \in I$ by

$$U(t,s) = \Gamma_t (1-\mathcal{G}_s\mathcal{V})^{-1} T_s. \tag{3.5}$$

The proof of Theorem 1 of [24] shows that $\{U(t,s)\colon t,s \in I\}$ is a strongly continuous family of unitary operators in $\mathcal{H}$ and it satisfies Eqn. (1.3) whenever $t,s,r \in I$. It is then well known (see e.g. [16]) that such a family can be patched together to produce a globally defined strongly continuous family of unitary operators $\{U(t,s)\colon t,s \in \mathbb{R}\}$ in $\mathcal{H}$ which satisfies (1.3). Then, property (1) of Theorem 1.4 is evidently satisfied.

We prove (2). It suffices to prove it when $t,s \in I$ for sufficiently small intervals I and u satisfies (3.2). By virtue of Theorem 2.1 and (2.7), $u_0(t) = U_0(t,s)f \in C^1(\mathbb{R},\Sigma(-2))$ and $i\dot{u}_0(t) = H_0(t)u_0(t)$. Sobolev embedding theorem implies

$$\Sigma(2) \subset \cap_D L_D^{l_D,2}(X) \text{ and, hence, } \Sigma(-2) \supset \textstyle\sum_D L_D^{l'_D,2}(X)$$

by duality. Thus, $\mathcal{V}u \in \mathcal{X}^*(I) \subset L^1(I,\Sigma(-2))$ and the function

$$g(t) = \int_s^t U_0(s,r)V(r)u(r)dr$$

is locally AC with values in $\Sigma(-2)$ and is simultaneously continuous with values in $\mathcal{H}$ by (2.21). It follows that

$$u(t) = u_0(t) + (\mathcal{G}_s\mathcal{V}u)(t) = u_0(t) - iU_0(t,s)g(t),$$

is $\Sigma(-2)$-valued locally AC and $i\dot{u}(t) = H_0(t)u(t) + V(t)u(t)$, a.e. t.

Finally, we show that any $u \in \mathcal{X}(I)$ which is locally AC with values in $\Sigma(-2)$ and which satisfies (1.2) with $u(s) = f \in \mathcal{H}$, $s \in I$ must do (3.2) and, hence, is unique. To see this, we first note that $(H_0(s)u, v) = (u, H_0(s)v)$ for $u \in \mathcal{H}$ and $v \in \Sigma(2)$ and that, for $w \in \Sigma(-2)$ and $\varphi \in \Sigma(2)$, $(w, U_0(s,t)\varphi) = (U_0(t,s)w, \varphi)$. This can be seen by approximating u and w respectively by sequences $u_n \in \Sigma(2)$ and $\omega_n \in \Sigma(2)$ such that $\|u_n - u\|_{\mathcal{H}} \to 0$ and $\|w_n - w\|_{\Sigma(-2)} \to 0$ as $n \to \infty$. Then, for $\varphi \in \Sigma(2)$ we have

$$\begin{aligned} i\frac{d}{ds}(u(s), U_0(s,t)\varphi) &= (H_0(s)u(s), U_0(s,t)\varphi) + (V(s)u(s), U_0(s,t)\varphi) \\ &\quad - (u(s), H_0(s)U_0(s,t)\varphi) = (U_0(t,s)V(s)u(s), \varphi), \quad \text{a.e.}\, s. \end{aligned} \tag{3.6}$$

It follows by integration that

$$(u(t), \varphi) - (U_0(t,0)u(0), \varphi) = -i\int_0^t (U_0(t,s)V(s)u(s), \varphi)ds$$

and $u(t)$ has to satisfy (3.2). This completes the proof. □

4 Proof of Theorem 1.8

We begin by recording several preliminaries for the proof. We assume in what follows that (φ, A) satisfies the conditions of Theorem 1.8.

4.1 Gauge transformation

We define time dependent gauge transformation

$$T(t)u(\underline{x}) = e^{itC\langle \underline{x}\rangle^2}u(\underline{x}), \quad (\mathcal{T}u)(t,\underline{x}) = T(t)u(\underline{x}), \quad t \in \mathbb{R}.$$

The following lemma is obvious.

Lemma 4.1. (1) *$\{T(t)\}$ is a strongly continuous unitary group in $\mathcal{H}$*

(2) *For any interval I, $\mathcal{T}$ is an isomorphism of the Banach space $\mathcal{X}(I)$ and at the same time of the space $C(I, \Sigma(2)) \cap C^1(I, L^2)$.*

(3) *$u(t,\underline{x})$ satisfies* (1.2) *if and only if $v(t,\underline{x}) = (\mathcal{T}u)(t,\underline{x})$ does the same with $\tilde{A}(t,x) = A(t,x) - 2tCx$ and $\tilde{\varphi}(t,x) = \varphi(t,x) + C\langle x\rangle^2$ replacing A and φ respectively. New potentials $\tilde{A}(t,x)$ and $\tilde{\varphi}(t,x)$ satisfy the conditions of* Theorem 1.8.

It follows from Lemma 4.1 (3) that

$$i\partial_t u = \tilde{H}_0(t)u + V(t,x)u \tag{4.1}$$

$$\tilde{H}_0(t) = -\sum_{j=1}^N \left(\frac{1}{2m_j}(\nabla_j - ie_j\tilde{A}(t,x_j))^2 + e_j\tilde{\varphi}(t,x_j)\right)u, \tag{4.2}$$

generates a unique unitary propagator which satisfies the properties of Theorem 1.4, which we denote by $\tilde{U}(t,s)$. Then, the uniqueness result of the theorem implies

$$U(t,s) = T(t)\tilde{U}(t,s)T(s)^{-1}$$

and we may prove Theorem 1.8 additionally assuming $\varphi(t,x) \geq C\langle x\rangle^2$, which we do in what follows. The merit of doing so is that $H_0(t)$ will then become selfadjoint with common domain $\Sigma(2)$ and with a core $C_0^\infty(X)$, see below.

For the proof of the next lemma we use the following well-known results on the n-dimensional quantum harmonic oscillator, $n = 1, 2, \dots$:

$$H_{os} = -\frac{1}{2}\Delta + \frac{1}{2}x^2 \quad x \in \mathbb{R}^n.$$

(a) H_{os} with domain $D(H_{os}) = \Sigma(2)$ is selfadjoint in $L^2(\mathbb{R}^n)$, $H_{os} \geq n/2$ and $C_0^\infty(\mathbb{R}^n)$ is a core.

(b) For any set of multi-indices α, β, γ and δ with $|\alpha+\beta+\gamma+\delta| \leq 2$, the operator $x^\alpha \partial^\beta H_{os}^{-1} x^\gamma \partial^\delta$ has a bounded extension in $\mathcal{H}$.

(c) The integral kernel $G(x,y)$ of H_{os}^{-1} satisfies for constants $C, c > 0$,

$$0 < G(x,y) \leq \frac{Ce^{-c|x-y|(1+|x|+|y|)}}{|x-y|^{n-2}}, \quad x, y \in \mathbb{R}^n. \tag{4.3}$$

We write $\partial_t u = \dot{u}$ and etc. hereafter.

Lemma 4.2. *Suppose that A and φ satisfy the assumption of* Theorem 1.8 *and that $\varphi(t,x) \geq \frac{1}{2}\langle x\rangle^2$, $(t,x) \in I_0 \times \mathbb{R}^d$. Then:*

(1) *The operator $H_0(t)$ with domain $D(H_0(t)) = \Sigma(2)$ is selfadjoint in $\mathcal{H}$, $H_0(t) \geq (Nd+1)/2$ and $C_0^\infty(X)$ is a core.*

(2) *$H_0(t)$ and $H_0(t)^{-1}$ are strongly differentiable functions of $t \in I_0$ with values in $\mathbf{B}(\Sigma(2),\mathcal{H})$ and $\mathbf{B}(\mathcal{H},\Sigma(2))$ respectively.*

Proof. We may assume $e_j = m_j = 1$, $1 \leq j \leq N$. For proving (1), we may also freeze t and we omit the variable t. We write $\|u\|_{\mathcal{H}} = \|u\|$. Since A and φ are smooth and φ is bounded below it is well known ([13, 11]) that H_0 on $C_0^\infty(X)$ is essentially selfadjoint; if we write the selfadjoint extension by the same letter, then H_0 is the maximal operator:

$$D(H_0) = \{u \in \mathcal{H} \colon H_0 u \in \mathcal{H}\}$$

where H_0 on the right hand side should be understood in the sense of distributions. It then is obvious that $\Sigma(2) \subset D(H_0)$. We prove the opposite inclusion and $H_0 \geq (Nd+1)/2$. Let $u \in D(H_0)$ and take a sequence $u_n \in C_0^\infty(\mathbb{R}^n)$ such that $\|u_n - u\| \to 0$ and $\|H_0 u_n - H_0 u\| \to 0$ as $n \to \infty$. Then, $|\partial_j |u_n|| \leq |(\partial_j - iA_j)u_n|$ and property (a) above imply

$$(H_0 u_n, u_n) \geq \frac{1}{2}\|\nabla |u_n|\|^2 + \frac{1}{2}\|\langle \underline{x}\rangle^2 u_n\|^2 \geq \left(\frac{Nd+1}{2}\right)\|u_n\|^2.$$

Letting $n \to \infty$, we obtain $H_0 \geq (Nd+1)/2$. Set $f = H_0 u$ and $f_n = H_0 u_n$. Then, Kato's inequality implies in the sense of distributions

$$\left(-\frac{1}{2}\Delta + \frac{1}{2}\langle \underline{x}\rangle^2\right)|u_n| \geq |f_n|.$$

Then, the property (c) above of the harmonic oscillator implies

$$|u_n(\underline{x})| \leq \left(H_{os} + \frac{1}{2}\right)^{-1} |f_n| \leq 2\langle \underline{x}\rangle^{-2}|f_n|(\underline{x}). \tag{4.4}$$

Expanding $(-i\nabla_j - A(x_j))^2$ in $f_n = H_0 u_n$, we see that

$$\left(-\frac{1}{2}\Delta + \frac{1}{2}\underline{x}^2\right) u_n = f_n + i\sum_{j=1}^{N} A(x_j)\cdot\nabla_j u_n + O(\underline{x}^2)u_n. \tag{4.5}$$

We want to show that

$$\|A(x_j)\cdot\nabla_j u_n\| \leq C\|f_n\|, \quad j = 1,\dots,N, \tag{4.6}$$

which, by virtue of (4.5) and (4.4), will imply

$$\|H_{os}u_n\| \leq \|f_n\| + C\|f_n\| + \|O(\underline{x}^2)u_n\| \leq C\|f_n\|.$$

Then, the standard argument will imply $u \in D(H_{os})$ and $\|H_{os}u\| \leq C\|f\|$ and, complete the proof of statement (1). To show (4.6), we solve (4.5) for u_n and apply $\partial_{jk}x_{jl}$, $k, l = 1,\dots,d$ to the resulting equation:

$$u_n = H_{os}^{-1}\left(f_n + i\sum_{j=1}^{N} A(x_j)\cdot\nabla_j u_n + O(\underline{x}^2)u_n\right). \tag{4.7}$$

Then, it is clear from property (b) of H_{os} and (4.4) that

$$\|\partial_{jk}x_{jl}H_{os}^{-1}(f_n + O(\underline{x}^2)u_n)\| \leq C\|f_n + O(\underline{x}^2)u_n\| \leq C\|f_n\|.$$

In the right of

$$\begin{aligned}\partial_{jk}x_{jm}H_{os}^{-1}\partial_{ln}A_n(x_l)u_n &= \partial_{jk}H_{os}^{-1}[H_{os}, x_{jm}]H_{os}^{-1}\partial_{ln}A_n(x_l)u_n \\ &\quad + \partial_{jk}H_{os}^{-1}[x_{jm},\partial_{ln}]A_n(x_l)u_n + \partial_{jk}H_{os}^{-1}\partial_{ln}x_{jm}A_n(x_l)u_n\end{aligned}$$

we have $[H_{os}, x_{jm}] = -\partial_{jm}$ and $[x_{jm}, \partial_{ln}] = -\delta_{jm,ln}$. It follows by virtue of property (b) of H_{os}^{-1} and (4.4) again that $\|\partial_{jk}x_{jm}H_{os}^{-1}\partial_{ln}A_n(x_l)u_n\| \leq C\|f_n\|$, hence $\|x_{jm}\partial_{jk}H_{os}^{-1}\partial_{ln}A_n(x_l)u_n\| \leq C\|f_n\|$. The desired estimate (4.6) follows evidently. This proves the statement (1).

By the assumptions on A and φ, $t \to H_0(t)u \in \mathcal{H}$ for $u \in \Sigma(2)$ is differentiable and

$$\dot{H}_0(t)u = \left(i\sum_{j=1}^{N}\dot{A}(t,x_j)\nabla_j + i\frac{1}{2}\mathrm{div}_j\,\dot{A}(t,x_j) + A(t,x_j)\dot{A}(t,x_j) + \dot{\varphi}(t,x_j)\right)u$$

is continuous with values in $\mathcal{H}$. Statement (2) is now obvious . $\square$

4.2 Proof of Theorem 1.8

The proof is an improvement of that of Theorem 7 of [24]. Define for compact intervals I the pair of function spaces

$$\mathcal{Y}(I) = \{u \in C(I, \Sigma(2)) \colon \dot{u} \in \mathcal{X}(I)\}, \tag{4.8}$$

$$\mathcal{Y}^*(I) = \{u \in C(I, \mathcal{H}) \colon \dot{u} \in \mathcal{X}^*(I)\}. \tag{4.9}$$

They are Banach spaces with natural norms

$$\|u\|_{\mathcal{Y}(I)} = \|u\|_{C(I,\Sigma(2))} + \|\dot{u}\|_{\mathcal{X}(I)}, \tag{4.10}$$

$$\|u\|_{\mathcal{Y}^*(I)} = \|u\|_{C(I,\mathcal{H})} + \|\dot{u}\|_{\mathcal{X}^*(I)}. \tag{4.11}$$

The following identities will be frequently used in what follows.

Lemma 4.3. *For $f \in \Sigma(2)$ and $f \in \mathcal{H}$ respectively, we have identities*

$$H_0(t)U_0(t,s)f = U_0(t,s)H_0(s)f + \int_s^t U_0(t,r)\dot{H}_0(r)U_0(r,s)f dr. \tag{4.12}$$

$$U_0(t,s)H_0(s)^{-1}f = H_0(t)^{-1}U_0(t,s)f - \int_s^t U_0(t,r)\left(\frac{d}{dr}H_0(r)^{-1}\right)U_0(r,s)f dr. \tag{4.13}$$

Proof. Let $f, g \in \mathcal{S}(X)$. Since $H_0(t)$ is selfadjoint in $\mathcal{H}$, we have

$$\frac{d}{dr}(H_0(r)U_0(r,s)f, U_0(r,t)g) = (\dot{H}_0(r)U_0(r,s)f, U_0(r,t)g).$$

Integrate both sides by r from s to t and obtain

$$(H_0(t)U_0(t,s)f, g) - (H_0(s)f, U_0(s,t)g) = \int_s^t (\dot{H}_0(r)U_0(r,s)f, U_0(r,t)g)dr,$$

which proves (4.12) on $\mathcal{S}(X)$. Since both sides of (4.12) is bounded from $\Sigma(2)$ to $\mathcal{H}$, it holds for $f \in \Sigma(2)$. Proof for (4.13) is similar and is omitted. □

Lemma 4.4. *Let $f \in \Sigma(2)$ and $u_0(t) = U_0(t,s)f$. Then, for any compact interval $I \subset I_0$, $u_0 \in \mathcal{Y}(I)$ and $\|u_0\|_{\mathcal{Y}(I)} \le C\|f\|_{\Sigma(2)}$ for a $C > 0$ independent of f and I.*

Proof. As was remarked under (2.9) that $u_0 \in C(\mathbb{R}, \Sigma(2))$. By virtue of (4.12), we have

$$i\frac{d}{dt}u_0(t) = U_0(t,s)H_0(s)f + \int_s^t U_0(t,r)\dot{H}_0(r)U_0(r,s)f dr \equiv u_1(t) + u_2(t).$$

Then, (2.20) implies $\|u_1\|_{\mathcal{X}(I)} \le C\|H_0(s)f\|_{\mathcal{H}} \le C\|f\|_{\Sigma(2)}$. By virtue of Lemma 4.2 (2), $\dot{H}_0(r)U_0(r,s)f$ is a continuous function of r with values in $\mathcal{H}$ and $\|\dot{H}_0(r)U_0(r,s)f\|_{L^1(I,\mathcal{H})} \le C\|f\|_{\Sigma(2)}$. It follows from (2.26) that

$$\|u_2\|_{\mathcal{X}(I)} \le C\|f\|_{\Sigma(2)}.$$

This proves the lemma. □

Recall that $\mathcal{G}_s$ is the integral operator defined by (3.1).

Lemma 4.5. *Let $s \in I_0$ and $I = [s-L, s+L] \subset I_0$. Then, $\mathcal{G}_s \in \mathbf{B}(\mathcal{Y}^*(I), \mathcal{Y}(I))$ and, for a constant $C > 0$ independent of s and L*

$$\|\mathcal{G}_s u\|_{\mathcal{Y}(I)} \le C\|u\|_{\mathcal{Y}^*(I)}, \quad u \in \mathcal{Y}^*(I). \tag{4.14}$$

Proof. It suffices to show the following two estimates:

$$\|\mathcal{G}_s u\|_{C(I,\Sigma(2))} \le C\|u\|_{\mathcal{Y}^*(I)}, \quad \|\dot{\mathcal{G}}_s u\|_{\mathcal{X}(I)} \le C\|u\|_{\mathcal{Y}^*(I)}. \tag{4.15}$$

Let $u \in \mathcal{Y}^*(I)$. Then $u(r)$ is AC on I with values in $\Sigma(-2)$ and $H_0(r)^{-1}$ is C^1 with values in $\mathbf{B}(\Sigma(-2), \mathcal{H})$ by virtue of Lemma 4.2 (2). It follows that $U_0(t,r)H_0(r)^{-1}u(r)$ is AC with respect to $r \in I$ with values in $\mathcal{H}$. We differentiate this function and integrate by r from s to t. This gives the following integration by parts formula:

$$\begin{aligned}
\mathcal{G}_s u(t) &= -\int_s^t (\partial_r U_0(t,r))H_0(r)^{-1}u(r)dr \\
&= U_0(t,s)H_0(s)^{-1}u(s) - H_0(t)^{-1}u(t) && (4.16)\\
&+ \int_s^t U_0(t,r)(\partial_r H_0(r)^{-1})u(r)dr + \int_s^t U_0(t,r)H_0(r)^{-1}\dot{u}(r)dr. && (4.17)
\end{aligned}$$

Since $H_0(t)^{-1} \in \mathbf{B}(\mathcal{H}, \Sigma(2))$ is strongly C^1 function of t by (2) of Lemma 4.2, it is obvious that two terms on (4.16) and the first integrals are $\Sigma(2)$-valued continuous functions of $t \in I$ and their norm in $C(I, \Sigma(2))$ are bounded by $C\|u\|_{C(I,\mathcal{H})} \le C\|u\|_{\mathcal{Y}^*(I)}$. Define for $u \in \mathcal{Y}^*(I)$

$$w(t) = \int_s^t U_0(t,r)\dot{u}(r)dr. \tag{4.18}$$

Then, Lemma 2.3 implies

$$w(t) \in \mathcal{X}(I), \quad \|w\|_{\mathcal{X}(I)} \le C\|\dot{u}\|_{\mathcal{X}^*(I)} \le C\|u\|_{\mathcal{Y}^*(I)}. \tag{4.19}$$

In particular $\|w\|_{C(I,\mathcal{H})} \le C\|u\|_{\mathcal{Y}^*(I)}$. If we use (4.13) and change the order of integration, then

$$\begin{aligned}
&\int_s^t U_0(t,r)H_0(r)^{-1}\dot{u}(r)dr = H_0(t)^{-1}\int_s^t U_0(t,r)\dot{u}(r)dr && (4.20)\\
&- \int_s^t U_0(t,\rho)\partial_r(H_0(\rho)^{-1})\left(\int_s^\rho U_0(\rho,r)\dot{u}(r)dr\right)d\rho && (4.21)\\
&= H_0(t)^{-1}w(t) - \int_s^t U_0(t,\rho)(\partial_r H_0(r)^{-1})w(\rho)d\rho. && (4.22)
\end{aligned}$$

Since $w \in C(I, \mathcal{H})$, Lemma 4.2 implies that two functions on (4.22) are both in $C(I, \Sigma(2))$ and are bounded by $C\|w\|_{C(I,\mathcal{H})}$. Thus first of (4.15) is proved.

We next prove the second of (4.15). Differentiating (4.16) and (4.17), we have

$$i\frac{d}{dt}\mathcal{G}_s u(t) = H_0(t)U_0(t,s)H_0^{-1}(s)u(s) + \int_s^t H_0(t)U_0(t,r)H_0(r)^{-1}\dot{u}(r)dr$$
$$- \int_s^t H_0(t)U_0(t,r)H_0(r)^{-1}\dot{H}_0(r)H_0(r)^{-1}u(r)dr \equiv a(t)+b(t)-c(t), \quad (4.23)$$

where the definition of $a(t), b(t)$ and $c(t)$ should be obvious. We rewrite these functions by using (4.12) and (4.13) in such a way that propagators $U_0(t,s)$ or $U(t,r)$ are placed on the left most or in front of $\dot{u}(r)$. Using (4.12) we rewrite:

$$a(t) = U_0(t,s)u(s)$$
$$+ \int_s^t U_0(t,r)\dot{H}_0(r)U_0(r,s)H_0(s)^{-1}u(s)dr = v_1(t)+v_2(t). \quad (4.24)$$

Lemma 2.3 implies $\|v_1\|_{\mathcal{X}(I)} \le C\|u\|_{C(I,\mathcal{H})} \le C\|u\|_{\mathcal{Y}^*(I)}$; Lemma 4.2 (2) implies $I \ni r \to \dot{H}_0(r)U_0(r,s)H_0(s)^{-1}u(s) \in \mathcal{H}$ is continuous and bounded by $C\|u\|_{\mathcal{Y}^*(I)}$. It follows via Lemma 2.5 that $\|v_2\|_{\mathcal{X}(I)} \le C\|u\|_{\mathcal{Y}^*(I)}$. Hence

$$\|a\|_{\mathcal{X}(I)} \le C\|u\|_{\mathcal{Y}^*(I)}. \quad (4.25)$$

We rewrite $b(t)$ by using (4.12) to place $U_0(t,r)$ in the front, yielding

$$b(t) = \int_s^t U_0(t,r)\dot{u}(r)dr$$
$$+ \int_s^t \left(\int_r^t U_0(t,\rho)\dot{H}_0(\rho)U_0(\rho,r)d\rho\right) H_0(r)^{-1}\dot{u}(r)dr = v_3(t)+v_4(t).$$

We have $v_3(t) = w(t)$ and $\|v_3\|_{\mathcal{X}(I)} \le C\|u\|_{\mathcal{Y}^*(I)}$. Changing the order of integration yields

$$v_4(t) = \int_s^t U_0(t,\rho)\dot{H}_0(\rho)\left(\int_s^\rho U_0(\rho,r)H_0(r)^{-1}\dot{u}(r)dr\right)d\rho.$$

Rewrite $U_0(\rho,r)H_0(r)^{-1}\dot{u}(r)$ in the inner integral via (4.13) and change the order of integration in the resuting equation. We obtain

$$v_4(t) = \int_s^t U_0(t,\rho)\dot{H}_0(\rho)H_0(\rho)^{-1}\left(\int_s^\rho U_0(\rho,r)\dot{u}(r)dr\right)d\rho$$
$$- \int_s^t U_0(t,\rho)\dot{H}_0(\rho)\left\{\int_s^\rho U_0(\rho,\gamma)\partial_\gamma(H_0(\gamma)^{-1})\left(\int_s^\gamma U_0(\gamma,r)\dot{u}(r)dr\right)d\gamma\right\}d\rho$$
$$= \int_s^t U_0(t,\rho)\dot{H}_0(\rho)H_0(\rho)^{-1}w(\rho)d\rho$$
$$- \int_s^t U_0(t,\rho)\dot{H}_0(\rho)\left(\int_s^\rho U_0(\rho,\gamma)\partial_\gamma(H_0(\gamma)^{-1})w(\gamma)d\gamma\right)d\rho.$$

By virtue of (4.19) and Lemma 4.2 (2), integrands on the right are both continuous functions of ρ with values in $\mathcal{H}$ and satisfy

$$\|\dot{H}_0(\rho)H_0(\rho)^{-1}w(\rho)\|_{C(I,\mathcal{H})} \le C\|u\|_{\mathcal{Y}^*(I)},$$
$$\left\|\dot{H}_0(\rho)\left(\int_s^\rho U_0(\rho,\gamma)\partial_\gamma(H_0(\gamma)^{-1})w(\gamma)d\gamma\right)\right\|_{C(I,\mathcal{H})} \le C\|u\|_{\mathcal{Y}^*(I)}.$$

Then, (2.26) once more produces $\|v_4\|_{\mathcal{X}(I)} \le C\|u\|_{\mathcal{Y}^*(I)}$ and we obtain

$$\|b\|_{\mathcal{X}(I)} \le C\|u\|_{\mathcal{Y}^*(I)}. \tag{4.26}$$

Finally, we estimate $c(t)$. We first rewrite $H_0(t)U_0(t,r)$ by using (4.12). After changing the order of integration, we have

$$\begin{aligned} c(t) &= \int_s^t U_0(t,r)\dot{H}_0(r)H_0(r)^{-1}u(r)dr \\ &+ \int_s^t U_0(t,\rho)\left(\int_s^\rho \dot{H}_0(\rho)U_0(\rho,r)H_0(r)^{-1}\dot{H}_0(r)H_0(r)^{-1}u(r)dr\right)d\rho. \end{aligned}$$

Lemma 4.2 (2) implies that integrands on the right are $\mathcal{H}$-valued continuous functions of r and ρ respectively and

$$\|\dot{H}_0(r)H_0(r)^{-1}u(r)\|_{C(I,\mathcal{H})} \le C\|u\|_{\mathcal{Y}^*(I)},$$
$$\left\|\int_s^\rho \dot{H}_0(\rho)U_0(\rho,r)H_0(r)^{-1}\dot{H}_0(r)H_0(r)^{-1}u(r)dr\right\|_{C(I,\mathcal{H})} \le C\|u\|_{\mathcal{Y}^*(I)}.$$

It follows from Lemma 2.5 that

$$\|c\|_{\mathcal{X}(I)} \le C\|u\|_{\mathcal{Y}^*(I)}. \tag{4.27}$$

Combining (4.25), (4.26) and (4.27) with (4.23), we obtain the second estimate of (4.15), completing the proof of the lemma. □

Lemma 4.6. *Suppose V satisfies* Assumption 1.6. *Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for intervals of $|I| < \delta$ there exists a constant $C_{\varepsilon,I}$ such that*

$$\|\mathcal{V}u\|_{\mathcal{Y}^*(I)} \le \varepsilon\|u\|_{\mathcal{Y}(I)} + C_{\varepsilon,I}\|u\|_{C(I,\mathcal{H})}, \quad u \in \mathcal{Y}(I). \tag{4.28}$$

Proof. By virtue of (3.3) and the argument at the beginning of the proof of Theorem 1.4, there exists $\delta > 0$ such that we have for intervals of $|I| < \delta$ that

$$\|\mathcal{V}\dot{u}\|_{\mathcal{X}^*(I)} \le C\max_D \|V_D\|_{\mathcal{I}_D^{cont,p_D}(I)}\|\dot{u}\|_{\mathcal{X}(I)} \le \varepsilon\|\dot{u}\|_{\mathcal{X}(I)}. \tag{4.29}$$

If we write $\partial_t V_D = W_1 + W_2 \in L^{b_D}(I, L^{q_D}(X_{D,r})) + L^1(I, L^\infty(X_{D,r}))$ as in Assumption 1.6, then Sobolev embedding theorem and Hölder's inequality imply that for sufficiently small $\delta > 0$

$$\|W_1 u\|_{L^{\theta'_D}(I,L_D^{l'_D,2})} \le \|W\|_{L^{b_D}(I,L^{q_D}(X_{D,r}))}\|u\|_{C(I,\Sigma(2))} < \varepsilon\|u\|_{\mathcal{Y}(I)}. \tag{4.30}$$

$$\|W_2u\|_{L^1(I,L^2(X))} \le \|W_2\|_{L^1(I,L^\infty(X_{D,r}))}\|u\|_{C(I,\Sigma(2))} < \varepsilon\|u\|_{\mathcal{Y}(I)}. \tag{4.31}$$

Here we used that indices satisfy

$$\frac{1}{q_D} + \left(\frac{1}{2} - \frac{2}{n_D}\right) = \frac{1}{l'_D}, \quad \frac{1}{\theta'_D} = \frac{1}{b_D}.$$

Combination of (4.29), (4.30) and (4.31) proves that, when $|I| < \delta$,

$$\|(d/dt)(\mathcal{V}u)\|_{\mathcal{X}^*(I)} \le \varepsilon\|u\|_{\mathcal{Y}(I)}. \tag{4.32}$$

When $V_D \in \mathcal{I}_D^{cont,\tilde{p}_D}(I)$, it is obvious that $V_D u \in C(I,\mathcal{H})$ for $u \in C(I,\Sigma(2))$. We want show that

$$\|V_D(t,x_{D,r})u\|_{C(I,\mathcal{H})} \le \varepsilon\|u\|_{C(I,\Sigma(2))} + C_\varepsilon\|u\|_{C(I,\mathcal{H})} \tag{4.33}$$

for all $D \subset \{1,\dots,N\}$. The argument at the beginning of the proof of Theorem 1.4 once more shows that, for any $\varepsilon > 0$, we may write $V_D \in \mathcal{I}_D^{cont,\tilde{p}_D}(I)$ as

$$V_D = V_D^{(1)} + V_D^{(2)}, \text{ with } \|V_D^{(1)}\|_{C(I,L^{\tilde{p}_D}(X_{D,r}))} \le \varepsilon.$$

Then, recalling that $\tilde{p} \ge 2$ if $n_D = 3$, we obtain (4.33) by using Hölder's inequality and the Sobolev embedding theorem. Thus, $\|\mathcal{V}u\|_{C(I,\mathcal{H})} \le \varepsilon\|u\|_{\mathcal{Y}(I)} + C_\varepsilon\|u\|_{C(I,\mathcal{H})}$. This with (4.32) proves the lemma. □

Completion of the proof of Theorem 1.8 We let $f \in \Sigma(2)$ and $u(t)$ be the solution of the integral equation (3.2):

$$u(t) = u_0(t) + (\mathcal{G}_s\mathcal{V}u)(t), \quad u_0(t) = U_0(t,s)f. \tag{4.34}$$

It suffices to show that, when $L > 0$ is sufficiently small, $u \in \mathcal{Y}(I)$ for $I = [s-L, s+L]$. By virtue of Lemma 4.5 and Lemma 4.6, we may take L small so that

$$\|\mathcal{G}_s\mathcal{V}\|_{\mathcal{X}(I)} < 1/2, \quad \|\mathcal{G}_s\mathcal{V}u\|_{\mathcal{Y}(I)} \le (1/2)\|u\|_{\mathcal{Y}(I)} + C\|u\|_{\mathcal{X}(I)}. \tag{4.35}$$

By virtue of Lemma 4.4 we have $u_0 \in \mathcal{Y}(I)$ and, if we successively define

$$u_n(t) = u_0(t) + \mathcal{G}_s\mathcal{V}u_{n-1}, \quad n = 1,2,\dots,$$

then, (4.35) implies $u_n \in \mathcal{X}(I) \cap \mathcal{Y}(I)$ for $n = 1,2,\dots$ and

$$\|u_n - u_{n-1}\|_{\mathcal{X}(I)} \le 2^{-n}\|u_0\|_{\mathcal{X}(I)}, \tag{4.36}$$

$$\|u_{n+1} - u_n\|_{\mathcal{Y}(I)} \le 2^{-1}\|u_n - u_{n-1}\|_{\mathcal{Y}(I)} + C\|u_n - u_{n-1}\|_{\mathcal{X}(I)}. \tag{4.37}$$

It follows that u_n converges to the solution $u \in \mathcal{X}(I)$ and that

$$\begin{aligned}\|u_{n+1} - u_n\|_{\mathcal{Y}(I)} &\le (1/2)\|u_n - u_{n-1}\|_{\mathcal{Y}(I)} + C2^{-n}\|u_0\|_{\mathcal{X}(I)} \\ &\le 2^{-n}\|u_1 - u_0\|_{\mathcal{Y}(I)} + Cn2^{-n}\|u_0\|_{\mathcal{X}(I)}.\end{aligned}$$

Thus, u_n converges to u also in $\mathcal{Y}(I)$. This proves $u \in C(I,\Sigma(2)) \cap C^1(I,\mathcal{H})$. The rest of the theorem immediately follows from this and the proof of Theorem 1.8 is completed. □

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